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CONJECTURES ON THE CHARACTER DEGREES OF THE HARADA–NORTON SIMPLE GROUP HN

BY

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ABSTRACT

We classify the radical subgroups and chains of the Harada–Norton simple group HN and verify the Alperin weight conjecture and the refined Dade conjecture due to Uno for the group. This implies the Isaacs–Navarro and Dade reductive conjectures for the group.

1. Introduction

Applying the local subgroup strategy of [2] and [3], we have previously classified the radical subgroups and radical chains for the sporadic simple groups Fi_{22}, Fi_{23}, Co_2, O'N, Ru and Co_1, and verified the Alperin and Dade reductive conjectures for these groups, (see [2], [3], [4], [5] and [6]). In this paper, we use the strategy to verify the Alperin weight conjecture and the refined Dade conjecture due to Uno for the Harada–Norton simple group HN. This implies the Isaacs–Navarro and Dade reductive conjectures for HN. The principal challenge is to construct the maximal p-local subgroups of HN.

Let G be a finite group, p a prime and B a p-block of G. Alperin [1] conjectured that the number of B-weights equals the number of irreducible Brauer characters of B. Dade [11] generalized the Knörr–Robinson version of the Alperin weight conjecture and presented his ordinary conjecture exhibiting the number of ordinary irreducible characters of a fixed defect in B in terms of an alternating

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sum of related values for p-blocks of some p-local subgroups of G. Dade [13] announced that his reductive conjecture needs only to be verified for finite non-abelian simple groups; in addition, if a finite group has a trivial Schur multiplier and cyclic outer automorphism group, then the invariant conjecture is equivalent to the reductive conjecture. Recently, Isaacs and Navarro [17] proposed a new conjecture which is a refinement of the Alperin–McKay conjecture, and Uno [20] proposed an alternating sum version of the Isaacs–Navarro conjecture which is a refinement of the Dade conjecture.

We verify the Alperin weight conjecture and Uno's refinement of Dade's invariant conjecture for HN. This implies the Isaacs–Navarro conjecture and Dade's inductive (and augmented inductive) conjecture for HN, since the Schur multiplier of HN is trivial.

The paper is organized as follows. In Section 2, we fix notation, state the conjectures in detail and show that the refinement of the invariant conjecture holds for all blocks with cyclic defect groups. In Section 3, we recall our modified local strategy and explain how we applied it to determine the radical subgroups of HN. In Section 4, we classify the radical subgroups of HN up to conjugacy and verify the Alperin weight conjecture. In Section 5, we do some cancellations in the alternating sum of Dade's conjecture, and then determine radical chains (up to conjugacy) and their local structures, and in the last section, we verify the refined invariant conjecture of Dade for HN.

2. The conjectures

Let R be a p-subgroup of a finite group G. Then R is **radical** if $O_p(N(R)) = R$, where $O_p(N(R))$ is the largest normal p-subgroup of the normalizer $N(R) = N_G(R)$. Denote by $\mathrm{Irr}(G)$ the set of all irreducible ordinary characters of G, and let $\mathrm{Blk}(G)$ be the set of p-blocks, $B \in \mathrm{Blk}(G)$ and $\varphi \in \mathrm{Irr}(N(R)/R)$. The pair (R, φ) is called a B-**weight** if $\mathrm{d}(\varphi) = 0$ and $B(\varphi)^G = B$ (in the sense of Brauer), where $\mathrm{d}(\varphi) = \log_p(|G|_p) - \log_p(\varphi(1)_p)$ is the p-defect of φ and $B(\varphi)$ is the block of $N(R)$ containing φ. A weight is always identified with its G-conjugates. Let $\mathcal{W}(B)$ be the number of B-weights, and $\ell(B)$ the number of irreducible Brauer characters of B. Alperin conjectured that $\mathcal{W}(B) = \ell(B)$ for each $B \in \mathrm{Blk}(G)$.

Given a p-subgroup chain

$$(2.1) \qquad C\colon P_0 < P_1 < \cdots < P_n$$

of G, define $|C| = n$, $C_k : P_0 < P_1 < \cdots < P_k$, $C(C) = C_G(P_n)$, and

$$(2.2) \qquad N(C) = N_G(C) = N(P_0) \cap N(P_1) \cap \cdots \cap N(P_n).$$

The chain C is said to be **radical** if it satisfies the following two conditions:

(a) $P_0 = O_p(G)$ and (b) $P_k = O_p(N(C_k))$ for $1 \leq k \leq n$.

Denote by $\mathcal{R} = \mathcal{R}(G)$ the set of all radical p-chains of G.

Let E be an extension of G, $F = E/G$, $C \in \mathcal{R}(G)$, $\psi \in \mathrm{Irr}(N_G(C))$ and $N_E(C,\psi)$ the stabilizer of (C,ψ) in E. Then $N_F(C,\psi) = N_E(C,\psi)/N_G(C)$ is a subgroup of F. For a subgroup $U \leq F$, denote by $\mathrm{Irr}(N_G(C), B, d, U)$ the set of characters ψ in $\mathrm{Irr}(N_G(C))$ such that $\mathrm{d}(\psi) = d$, $B(\psi)^G = B$ and $N_F(C,\psi) = U$. Set $\mathrm{k}(N_G(C), B, d, U) = |\,\mathrm{Irr}(N_G(C), B, d, U)|$. In the notation above the Dade invariant conjecture is stated as follows.

Dade's Invariant Conjecture ([13]): *If $O_p(G) = 1$ and B is a p-block of G with defect group $D(B) \neq 1$, then for any integer $d \geq 0$,*

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|} \mathrm{k}(N_G(C), B, d, U) = 0$$

where $\mathcal{R}/G$ is a set of representatives for the G-orbits of $\mathcal{R}$.

Let H be a subgroup of a finite group G and $\varphi \in \mathrm{Irr}(H)$. The p-remainder $r(\varphi) = r_p(\varphi)$ of φ is the integer $0 < r(\varphi) \leq (p-1)$ such that the p'-part $(|H|/\varphi(1))_{p'}$ of $|H|/\varphi(1)$ satisfies

$$\left(\frac{|H|}{\varphi(1)}\right)_{p'} \equiv r(\varphi) (\mathrm{mod}\, p).$$

Given integer $1 \leq r \leq (p-1)/2$, let $\mathrm{Irr}(H, [r])$ be the subset of $\mathrm{Irr}(H)$ consisting of characters φ such that $r(\varphi) \equiv \pm r (\mathrm{mod}\, p)$, and let $\mathrm{Irr}(H, B, d, U, [r]) = \mathrm{Irr}(H, B, d, U) \cap \mathrm{Irr}(H, [r])$ and $\mathrm{k}(H, B, d, U, [r]) = |\,\mathrm{Irr}(H, B, d, U, [r])|$.

Let $B \in \mathrm{Blk}(G)$ with a defect group $D = D(B)$ and the Brauer correspondent $b \in \mathrm{Blk}(N_G(D))$. Then

$$\mathrm{k}(N_G(D), B, \mathrm{d}(B), [r]) = \sum_{U \leq F} \mathrm{k}(N_G(D), B, \mathrm{d}(B), U, [r])$$

is the number of characters $\varphi \in \mathrm{Irr}(b)$ such that φ has height 0 and $r(\varphi) \equiv \pm r (\mathrm{mod}\, p)$, where $\mathrm{d}(B)$ is the defect of B.

Isaacs–Navarro Conjecture ([17, Conjecture B]): *In the notation above,*

$$\mathrm{k}(G, B, \mathrm{d}(B), [r]) = \mathrm{k}(N_G(D), B, \mathrm{d}(B), [r]).$$

The following refinement of Dade's conjecture is due to Uno.

UNO'S CONJECTURE ([20, Conjecture 3.2]): *If $O_p(G) = 1$ and if $D(B) > 1$, then for any integers $d \geq 0$ and $1 \leq r \leq (p-1)/2$,*

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|} \mathrm{k}(N_G(C), B, d, U, [r]) = 0. \tag{2.3}$$

Note that if $p = 2$ or 3, then the conjecture is equivalent to Dade's invariant conjecture.

PROPOSITION 2.1: *Uno's refinement of the Dade invariant conjecture holds for all blocks with cyclic defect groups.*

Proof: By Dade [11, Corollary 3.12], we can replace the family of radical chains $\mathcal{R}$ by that consisting of elementary abelian chains $\mathcal{E}$ of G. Since $D = D(B) = \langle x \rangle$ is cyclic, it follows that $\mathcal{E}/G = \{1, 1 < \Omega(D)\}$, where $\Omega(D) \leq D$ is the unique subgroup of order p. Thus $N_G(1 < \Omega(D)) = N_G(\Omega(D)) = \tilde{G}$ and (2.3) is equivalent to

$$\mathrm{k}(G, B, d, U, [r]) = \mathrm{k}(\tilde{G}, B, d, U, [r]). \tag{2.4}$$

Following the notation of [12, Section 4], let $B_D \in \mathrm{Blk}(N_G(D))$ be the Brauer correspondent of B, and $\tilde{B}$ the block of $\tilde{G} = N_G(\Omega(D))$ with $B_D^{\tilde{G}} = \tilde{B}$. Then D is a defect group of $\tilde{B}$ and B_D is the Brauer correspondent of $\tilde{B}$.

By [17, Theorem (2.2)] there is a bijection $\chi \mapsto \psi = \psi_\chi$ of $\mathrm{Irr}(B)$ onto $\mathrm{Irr}(B_D)$ such that

$$\chi(xy) = \epsilon_\chi \psi(xy), \tag{2.5}$$

for any $y \in L_{p'}$, where $\epsilon_\chi = \pm 1$ and $L_{p'}$ is the set of p-regular elements of $L = C_G(x)$. As shown in the proof of [17, Theorem (2.1)] (2.5) implies that $\chi(1)_{p'} \equiv \pm |G: N_G(D)|_{p'} \psi(1)_{p'} (\operatorname{mod} p)$ and, in particular,

$$r(\chi) \equiv \pm r(\psi) \qquad (\operatorname{mod} p). \tag{2.6}$$

Replace G by $\tilde{G}$ and B by $\tilde{B}$. It follows that there exists a bijection $\Phi\colon \chi \mapsto \tilde{\chi}$ of $\mathrm{Irr}(B)$ onto $\mathrm{Irr}(\tilde{B})$ such that $r(\chi) \equiv \pm r(\tilde{\chi}) (\operatorname{mod} p)$.

Let $\mathrm{Irr}(B, [r]) = \mathrm{Irr}(G, [r]) \cap \mathrm{Irr}(B)$, $\chi \in \mathrm{Irr}(B, [r])$ and $\psi = \psi_\chi \in \mathrm{Irr}(B_D, [r])$. If $\tau \in N_F(B)$, then τ normalizes D and so $x' = x^{\tau^{-1}}$ is also a generator of D. Thus $y' = y^{\tau^{-1}} \in L_{p'}$ and (cf. [10, Corollary 1.9])

$$\chi^\tau(xy) = \chi(x'y') = \epsilon_\chi \psi(x'y') = \epsilon_\chi \psi^\tau(xy). \tag{2.7}$$

Similarly, $\tilde{\chi}^\tau$ and ψ^τ satisfy (2.7) with χ replaced by $\Phi(\chi) = \tilde{\chi}$. It follows by the definition of Φ that $\Phi(\chi^\tau) = \Phi(\chi)^\tau$, so that $\mathrm{Irr}(B,[r])$ and $\mathrm{Irr}(\tilde{B},[r])$ are $N_F(B)$-isomorphic. This implies (2.4). ∎

LEMMA 2.2: *Let $\sigma : O_p(G) < P_1 < \cdots < P_{m-1} < P_m = Q < P_{m+1} < \cdots < P_\ell$ be a fixed radical p-chain of a finite group G, where $1 \leq m < \ell$. Suppose*

$$\sigma' : O_p(G) < P_1 < \cdots < P_{m-1} < P_{m+1} < \cdots < P_\ell$$

is also a radical p-chain such that $N_G(\sigma) = N_G(\sigma')$ and $N_E(\sigma) = N_E(\sigma')$. Let $\mathcal{R}^-(\sigma, Q)$ be the subfamily of $\mathcal{R}(G)$ consisting of chains C whose $(\ell - 1)$-th subchain $C_{\ell-1}$ is conjugate to σ' in G, and $\mathcal{R}^0(\sigma, Q)$ the subfamily of $\mathcal{R}(G)$ consisting of chains C whose ℓ-th subchain C_ℓ is conjugate to σ in G. Then the map g sending any $O_p(G) < P_1 < \cdots < P_{m-1} < P_{m+1} < \cdots < P_\ell < \cdots$ in $\mathcal{R}^-(\sigma, Q)$ to $O_p(G) < P_1 < \cdots < P_{m-1} < Q < P_{m+1} < \cdots < P_\ell < \cdots$ induces a bijection, denoted again by g, from $\mathcal{R}^-(\sigma, Q)$ onto $\mathcal{R}^0(\sigma, Q)$. Moreover, for any C in $\mathcal{R}^-(\sigma, Q)$, we have $|C| = |g(C)| + 1$,

$$N_G(C) = N_G(g(C)) \quad \text{and} \quad N_E(C) = N_E(g(C)).$$

Proof: Straightforward. ∎

Suppose G is the Harada–Norton simple group and E is the automorphism group of G. Then E/G is cyclic of order 2, so that U is determined uniquely by its order $|U|$. We set

$$\mathrm{k}(N_G(C), B, d, U, [r]) = \mathrm{k}(N_G(C), B, d, |U|, [r]).$$

3. A local subgroup strategy

The maximal subgroups of HN were classified by Norton and Wilson [19]. Using this classification and its proof, we know that when $p = 2$, 3 or 5, there are respectively 5, 4 or 3 maximal p-local subgroups M up to conjugacy. Thus each radical p-subgroup R of HN is radical in one of the subgroups M and further $N_{\mathrm{HN}}(R) = N_M(R)$.

In [2] and [3], a (modified) local strategy was developed to classify the radical p-subgroups R. We review this method here. Suppose M is a subgroup of G satisfying $N_G(R) = N_M(R)$.

Step (1). We first consider the case where M is a p-local subgroup. Let $Q = O_p(M)$, so that $Q \leq R$. Choose a subgroup X of M. Using MAGMA, we explicitly compute the coset action of M on the cosets of X in M; we obtain a group W representing this action, a group homomorphism f from M to W, and the kernel K of f. For a suitable X, we have $K = Q$ and the degree of the action of W on the cosets is much smaller than that of M. We can now directly classify the radical p-subgroup classes of W, and the preimages in M of the radical subgroup classes of W are the radical subgroup classes of M.

Step (2). Now consider the case where M is not p-local. We may be able to find its radical p-subgroup classes directly. Alternatively, we find a subgroup K of M such that $N_K(R) = N_M(R)$ for each radical subgroup R of M. If K is p-local, then we apply Step (1) to K. If K is not p-local, we can replace M by K and repeat Step (2).

Steps (1) and (2) constitute the **modified local strategy**. After applying the strategy, we list subgroups R satisfying $N_M(R) = N_{\mathrm{HN}}(R)$, so these are the radical subgroups of HN. Possible fusions among the resulting list of radical subgroups can be decided readily by testing whether the subgroups in the list are pairwise HN-conjugate.

In our investigations of the conjecture for HN, we used the minimal degree representation of HN as a permutation group on 1140000 points. The maximal subgroups of HN were constructed using the details supplied in [9] and the black-box algorithms of Wilson [22]. We also made extensive use of the procedures described in [2] and [3] for deciding the conjectures.

The computations reported in this paper were carried out using MAGMA V2.8-3 [8] on a Sun UltraSPARC Enterprise 4000 server.

4. Radical subgroups and weights

Let $\Phi(G,p)$ be a set of representatives for conjugacy classes of radical p-subgroups of G. For $H, K \leq G$, we write $H \leq_G K$ if $x^{-1}Hx \leq K$; and write $H \in_G \Phi(G,p)$ if $x^{-1}Hx \in \Phi(G,p)$ for some $x \in G$. We follow the notation of [9]. In particular, if p is odd, then $p_+^{1+2\gamma}$ is an extra-special group of order $p^{1+2\gamma}$ with exponent p; if δ is $+$ or $-$, then $2_\delta^{1+2\gamma}$ is an extra-special group of order $2^{1+2\gamma}$ with type δ. If X and Y are groups, we use $X.Y$ and $X:Y$ to denote an extension and a split extension of X by Y, respectively. Given a positive integer n, we use E_{p^n} or simply p^n to denote the elementary abelian group of order p^n, $\mathbb{Z}_n$ or simply n to denote the cyclic group of order n, and D_{2n} to denote the dihedral group of order $2n$.

Let G be the Harada–Norton simple group HN. Then

$$|G| = 2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19,$$

and we may suppose $p \in \{2,3,5\}$, since both conjectures hold for a block with a cyclic defect group by [12, Theorem 7.11] and Proposition 2.1.

We denote by $\mathrm{Irr}^0(H)$ the set of ordinary irreducible characters of p-defect 0 of a finite group H and by $\mathrm{d}(H)$ the number $\log_p(|H|_p)$. Given $R \in \Phi(G,p)$, let $C(R) = C_G(R)$ and $N = N_G(R)$. If $B_0 = B_0(G)$ is the principal p-block of G, then (cf. (4.1) of [2])

$$\mathcal{W}(B_0) = \sum_R |\,\mathrm{Irr}^0(N/C(R)R)|, \tag{4.1}$$

where R runs over the set $\Phi(G,p)$ such that $\mathrm{d}(C(R)R/R) = 0$. The character table of $N/C(R)R$ can be calculated by MAGMA, and so we find $|\,\mathrm{Irr}^0(N/C(R)R)|$. If $\mathrm{d}(C(R)R/R) \neq 0$, then we leave the entries of the last column blank in Tables 1–2, since they do not contribute weights for the principal block.

PROPOSITION 4.1: *Let $G = \mathrm{HN}$. The non-trivial radical p-subgroups R of G (up to conjugacy) and their local structures are given in Tables 1 and 2 according as p is odd or even, where H^* denotes a subgroup of G such that $H^* \simeq H$ and $H^* \neq_G H$, Sy_p is a Sylow p-subgroup of G and SD_{16} is the semidihedral group of order 16. Moreover, if E is the automorphism group $\mathrm{Aut}(G)$ of G, then $N_E(R) = N(R).2$ for each radical subgroup R.*

R	$C(R)$	N	$\|\mathrm{Irr}^0(N/C(R)R)\|$
5	$5 \times U_3(5)$	$(D_{10} \times U_3(5)).2$	
$5^2.5_+^{1+2}$	5^2	$5^2.5_+^{1+2}.4A_5$	2
5_+^{1+4}	5	$5_+^{1+4}\colon 2_-^{1+4}.5.4$	6
Sy_5	5	$Sy_5.(2 \times 4)$	8
3	$3 \times A_9$	$(3 \times A_9)\colon 2$	
3^2	$3^2 \times A_6$	$(3^2\colon 4 \times A_6).2^2$	
3^4	3^4	$3^4 : 2(A_4 \times A_4).4$	4
$3^4.3$	3^2	$3^4.3.2S_4$	2
3_+^{1+4}	3	$3_+^{1+4}\colon 4A_5$	6
Sy_3	3	$Sy_3.(4 \times 2)$	8

Table 1. Non-trivial radical p-subgroups of HN with p odd

Proof: (1) Suppose $p = 5$. By [19, Section 3.3], HN has 3 maximal 5-subgroups, $M_1 = N(5A) = (D_{10} \times U_3(5)).2$, $M_2 = N(5B) = 5_+^{1+4}\colon 2_-^{1+4}.5.4$ and $M_3 = N(5B^2) = 5^2.5_+^{1+2} : 4A_5$.

Using MAGMA we first get a Sylow 5-subgroup $S = Sy_5$, then a faithful permutation representation ρ of S on 625 points, and then calculate the subgroup classes of $\rho(S)$. Thus we get all the subgroups of S, and so construct each maximal subgroup M_i.

Since a Sylow 5-subgroup Q_i of each M_i is the only radical 5-subgroup of M_i other than $O_5(M_i)$, it follows that the radical 5-subgroups of HN are as claimed.

R	$C(R)$	$N(R)$	$\lvert\mathrm{Irr}^0(N/C(R)R)\rvert$
2	$2.HS$	$2.HS.2$	
2^2	$2^2 \times A_8$	$(A_4 \times A_8) : 2$	
D_8	$2 \times A_6.2$	$D_8.A_6.2^2$	
Q_8	$2 \times 5{:}\ 4$	$(Q_8 \times 5{:}\ 4).S_3$	
SD_{16}	$2 \times 5{:}\ 2$	$(SD_{16} \times 5{:}\ 2).2$	
2^6	2^6	$2^6.U_4(2)$	1
2_+^{1+8}	2	$2_+^{1+8}(A_5 \times A_5).2$	0
$2_+^{1+8}.2$	2	$2_+^{1+8}.2A_5$	1
$2_+^{1+8}.2^2$	2	$2_+^{1+8}.2^2.(3 \times A_5)$	3
$2^3.2^2.2^6$	2^3	$2^3.2^2.2^6.(3 \times L_3(2))$	3
$2^2.2.2^2.2^4.2^4$	2^2	$2^2.2{\cdot}2^2.2^4.2^4.(3 \times S_3)$	3
$2_+^{1+8}.2^4$	2	$2_+^{1+8}.2^4.(3 \times S_3)$	3
Sy_2	2	$Sy_2.3$	3

Table 2. Non-trivial radical 2-subgroups of HN

(2) Suppose $p = 3$. As shown in [19, Section 3.2], HN has 4 maximal 3-local subgroups, $M_1 = N(3^2) = (3^2 : 4 \times A_6).2^2$, $M_2 = N(3^4) = 3^4{:}\ 2(A_4 \times A_4).4$, $M_3 = N(3B) = 3_+^{1+4} : 4A_5$ and $M_4 = N(3A) = (3 \times A_9) : 2$.

We use a method similar to the case when $p = 5$ to get all subgroup classes of a Sylow 3-subgroup, and then construct each maximal 3-local subgroup M_i.

Let R be a non-trivial radical 3-subgroup of G. Then $N(R)$ is 3-local, so that we may suppose $N(R) \leq M_i$ for some i and hence $R \in \Phi(M_i, 3)$ with $N(R) = N_{M_i}(R)$. We apply the local strategy of [2] or the modified local strategy [3] to each M_i.

If $M = M_1$ or M_3, then a Sylow subgroup of M is the only radical 3-subgroup of M other than $O_3(M)$. Thus we may take

$$\Phi(M_i, 3) = \begin{cases} \{3^2, 3^4\} & \text{if } i = 1, \\ \{3_+^{1+4}, Sy_3\} & \text{if } i = 3, \end{cases}$$

and in addition, $N_{M_1}(3^4) = 3^4.4^2.2^2$.

Let $3 = O_3(M_4)$ and S' a Sylow 3-subgroup of A_9. Then we may take

$$\Phi(M_4, 3) = \{3, 3^2, 3^3, 3^4, 3 \times S'\}.$$

Moreover, $N(R) \neq N_{M_4}(R)$ for each $R \in \Phi(M_4, 3) \backslash \{3\}$ and

$$N_{M_4}(R) = \begin{cases} 3.(S_3 \times S_6) & \text{if } R = 3^2, \\ 3^3 : 2S_4 & \text{if } R = 3^3, \\ 3^4 : (2 \times S_4) & \text{if } R = 3^4, \\ (3 \times S'){:}\ 2^2 & \text{if } R = 3 \times S'. \end{cases}$$

If $3^4 = O_3(M_2)$, then we may take

$$\Phi(M_2, 3) = \{3^4, 3^4.3, Sy_3\}$$

and, moreover, for $R \in \Phi(M_2, 3)$, $N_{M_2}(R) = N_{\mathrm{HN}}(R)$, so we may suppose $\Phi(M_2, 3) \subseteq \Phi(\mathrm{HN}, 2)$. This classifies the radical 3-subgroups of HN.

(3) Suppose $p = 2$. As shown in [19, Section 3.1] the maximal 2-local subgroups of HN are $M_1 = N(2A) = 2.HS.2$, $M_2 = N(2B) = 2_+^{1+8}.(A_5 \times A_5) : 2$, $M_3 = N(2^6) = 2^6.U_4(2)$, $M_4 = N(2B^3) = 2^3.2^2.2^6.(3 \times L_3(2))$ and $M_5 = N(2A^2) = (A_4 \times A_8) : 2$.

Using MAGMA we first fix a Sylow 2-subgroup S, then get a faithful permutation representation ρ of S on 128 points and then get the conjugacy classes of $\rho(S)$. Thus we can obtain the conjugacy classes of S, and construct the subgroups M_1 and M_2. Similarly, using $\rho(S)$ we can get all the normal subgroup classes of S and $2^3 = O_2(M_4)$ is the only elementary abelian normal subgroup of S of order 2^3, up to conjugacy. Thus we can construct M_4.

Suppose HN is given by the permutation representation on 1140000 points. Then the stabilizer of HN on any point is a maximal subgroup $M \simeq A_{12}$ of HN. Using a faithful representation of M on 12 points, we can construct the subgroups $M_5 \leq M$ and $2^6 = O_2(M_3) \leq M$, and so $M_3 = N(2^6)$.

Similarly, we may suppose each non-trivial radical 2-subgroup R of G is radical in some M_i with $N(R) \leq M_i$. We apply the local strategy of [2] or the modified local strategy [3] to each M_i.

(3.1) We may take

$$\Phi(M_1, 2) = \{2, 2^2, D_8, Q_8, SD_{16}, 2^6, 2^2.2^6, 2^4.2^4, 2^2.2^4.2^3, 2^2.2^3.2^5, 2^3.2^3.2^4, S'\},$$

and moreover, $N(R) \neq N_{M_1}(R)$ for precisely $R \in \Phi(M_1, 2) \backslash \{2, D_8, Q_8, SD_{16}\}$, where S' is a Sylow 2-subgroup of M_1. Note that if a non-abelian 2-group Q

has no non-cyclic normal abelian subgroup, then by [15, Theorem 5.4.10], Q is dihedral, semidihedral or generalized quaternion, and a generalized quaternion group has no non-cyclic abelian subgroup, and in addition, by [15, Theorem 5.4.3], a dihedral group has no quaternion subgroup. Moreover,

$$N_{M_1}(R) = \begin{cases} (2^2 \times A_8).2 & \text{if } R = 2^2, \\ 2^6.S_6 & \text{if } R = 2^6, \\ 2^2.2^6.S_5 & \text{if } R = 2^2.2^6, \\ 2^4.2^4.L_3(2) & \text{if } R = 2^4.2^4, \\ 2^2.2^4.2^3.S_3 & \text{if } R = 2^2.2^4.2^3, \\ 2^2.2^3.2^5.S_3 & \text{if } R = 2^2.2^3.2^5, \\ 2^3.2^3.2^4.S_3 & \text{if } R = 2^3.2^3.2^4, \\ S' & \text{if } R = S'. \end{cases}$$

(3.2) We may take

$$\Phi(M_2, 2) = \{2_+^{1+8}, 2_+^{1+8}.2, 2_+^{1+8}.2^2, 2_+^{1+8}.2^4, Sy_2\},$$

and moreover, $N(R) = N_{M_2}(R)$ for each $R \in \Phi(M_2, 2)$, so that we may suppose $\Phi(M_2, 2) \subseteq \Phi(G, 2)$.

(3.3) We may take

$$\Phi(M_3, 2) = \{2^6, 2^6.2^4, 2^2.2^3.2^6, S''\},$$

and moreover, $N(R) \neq N_{M_3}(R)$ for each $R \in \Phi(M_3, 2)\backslash\{2^6\}$, where S'' is a Sylow 2-subgroup of M_3. In addition,

$$N_{M_3}(R) = \begin{cases} 2^6.2^4.A_5 & \text{if } R = 2^6.2^4, \\ 2^2.2^3.2^6.3^2.2 & \text{if } R = 2^2.2^3.2^6, \\ S''.3 & \text{if } R = S''. \end{cases}$$

(3.4) If $2^3.2^2.2^6 = O_2(M_4)$, then we may take

$$\Phi(M_4, 3) = \{2^3.2^2.2^6, 2^2.2.2^2.2^4.2^4, 2_+^{1+8}.2^4, Sy_2\},$$

and moreover, $N(R) = N_{M_4}(R)$ for each $R \in \Phi(M_4, 2)$, so that we may suppose $\Phi(M_4, 2) \subseteq \Phi(G, 2)$.

(3.5) If $2^2 = O_2(M_5)$, then we may take

$$\Phi(M_5, 2) = \{2^2, D_8, 2^5, 2^6, 2^3.2^4, 2^4.2^3, (2^4.2^3)^*, 2^2.2^3.2^3, 2^3.2^2.2^3, 2^2.2^3.2^3.2\},$$

and moreover, $N(R) \neq N_{M_5}(R)$ for $R \in \Phi(M_5, 2)\backslash\{2^2\}$. In addition,

$$N_{M_5}(R) = \begin{cases} D_8 \times S_6 & \text{if } R = D_8, \\ A_4 \times 2^3 : L_3(2) & \text{if } R = 2^5, \\ 2^6.(3 \times S_3 \times S_3).2 & \text{if } R = 2^6, \\ 2^3.2^4.(S_3 \times S_3) & \text{if } R = 2^3.2^4, \\ 2^4.2^3.3.S_3 & \text{if } R = 2^4.2^3, \\ (2^4.2^3)^*.S_3 & \text{if } R = (2^4.2^3)^*, \\ 2^2.2^3.2^3.S_3 & \text{if } R = 2^2.2^3.2^3, \\ 2^3.2^2.2^3.S_3 & \text{if } R = 2^3.2^2.2^3, \\ 2^2.2^3.2^3.2 & \text{if } R = 2^2.2^3.2^3.2. \end{cases}$$

Thus the radical 2-subgroups are as claimed.

The centralizers and normalizers of R can be obtained by MAGMA, and by the local structures of $R \in \Phi(M_i, p)$, $N_E(R) = N_G(R).2$. ∎

LEMMA 4.2: *Let $G = \mathrm{HN}$ and $B_0 = B_0(G)$, and let $\mathrm{Blk}^+(G,p)$ be the set of p-blocks with a non-trivial defect group and $\mathrm{Irr}^+(G)$ the characters of $\mathrm{Irr}(G)$ with positive p-defect. If a defect group $D(B)$ of B is cyclic, then $\mathrm{Irr}(B)$ is given by* [16, p. 248].

(a) *If $p = 5$, then $\mathrm{Blk}^+(G,p) = \{B_0, B_1\}$ such that $D(B_1) \simeq 5$, so that $\mathrm{Irr}(B_0) = \mathrm{Irr}^+(G)\backslash \mathrm{Irr}(B_1)$. Moreover, $\ell(B_1) = 4$ and $\ell(B_0) = 16$.*

(b) *If $p = 3$, then $\mathrm{Blk}^+(G,p) = \{B_i \mid 0 \leq i \leq 2\}$ such that $D(B_1) \simeq 3^2$ and $D(B_2) \simeq 3$. In the notation of* [9, p. 164],

$$\mathrm{Irr}(B_1) = \{\chi_8, \chi_{10}, \chi_{19}, \chi_{32}, \chi_{33}, \chi_{37}, \chi_{43}, \chi_{49}, \chi_{50}\}$$

and $\mathrm{Irr}(B_0) = \mathrm{Irr}^+(G)\backslash(\bigcup_{i=1}^{2} \mathrm{Irr}(B_i))$. Moreover, $\ell(B_2) = 2$, $\ell(B_1) = 7$ and $\ell(B_0) = 20$.

(c) *If $p = 2$, then $\mathrm{Blk}^+(G,2) = \{B_0, B_1\}$ such that $D(B_1) \simeq SD_{16}$. In the notation of* [9, p. 164],

$$\mathrm{Irr}(B_1) = \{\chi_{17}, \chi_{34}, \chi_{35}, \chi_{36}, \chi_{37}, \chi_{44}, \chi_{45}, \chi_{49}\}$$

and $\mathrm{Irr}(B_0) = \mathrm{Irr}^+(G)\backslash \mathrm{Irr}(B_1)$. Moreover, $\ell(B_1) = 3$ and $\ell(B_0) = 17$.

Proof: If $B \in \mathrm{Blk}(G,p)$ is non-principal with $D = D(B)$, then $\mathrm{Irr}^0(C(D)D/D)$ has a non-trivial character θ and $N(\theta)/C(D)D$ is a p'-group, where $N(\theta)$ is the stabilizer of θ in $N(D)$. By [16, p. 248], we may suppose D is non-cyclic. Thus $D \in \{3^2, 2^2, D_8, Q_8, SD_{16}\}$. If $D = 2^2, D_8$ or Q_8, then $\mathrm{Irr}^0(C(D)D/D) = \{\theta\}$ and $|N(D) : C(D)D|_2 = 2$, so that there is a 2-element in $N(D)\backslash C(D)D$ stabilizing θ and D is not a defect group. Thus $D \in \{3^2, SD_{16}\}$. In the former case

$|\operatorname{Irr}^0(C(D)D/D)| = 1$ and in the latter case $|\operatorname{Irr}^0(C(D)D/D)| = 2$ with only one $N(D)$-orbit. It follows that in both cases G has exactly one block with defect D.

Using the method of central characters, $\operatorname{Irr}(B)$ is as above. If $D(B)$ is cyclic, then $\ell(B)$ is given by [16, p. 248].

If $p = 3$ and $B = B_1$, then $D(B) =_G 3^2$ and the non-trivial elements of $D(B)$ are of type $3A$, and $C_G(3A) = 3 \times A_9$. It follows by [18, Theorem 5.4.13] that $\mathrm{k}(B) = \ell(B) + \ell(b)$, where $b \in \operatorname{Blk}(3 \times A_9)$ such that each $b^G = B$. In addition, $b = B_0(3) \times b'$ with $b' \in \operatorname{Blk}(A_9)$ and $D(b') \simeq 3$, so that $\ell(b')$ is the number of b'-weights, which is 2 since $N_{A_9}(D(b')) = (3 \times A_6).2$. Thus $\ell(B) = 9 - 2 = 7$.

If $p = 2$ and $B = B_1$, then $D(B) =_G SD_{16}$ and, by [21], $\mathcal{W}(B) = \ell(B)$. Since HS has no irreducible character of 2-defect 0, it follows that there is no B-weight of the form $(2, \varphi)$. If $R = D_8$, then $N(D)/C(D)D$ has order 2 and $C(D)D/D$ contains a unique irreducible character θ of 2-defect 0. Thus θ has two extensions to $N(D)$ with positive 2-defect. So there is no B-weight of the form (R, φ). If $Q = 2^2, Q_8$ or SD_{16}, then there exists exactly one B-weight of the form (Q, φ), so that $\ell(B) = 3$.

If $\ell_p(G)$ is the number of p-regular G-conjugacy classes, then $\ell_5(G) = 24$, $\ell_3(G) = 38$ and $\ell_2(G) = 21$. Thus $\ell(B_0)$ can be calculated by the following equation due to Brauer:

$$\ell_p(G) = \sum_{B \in \operatorname{Blk}^+(G,p)} \ell(B) + |\operatorname{Irr}^0(G)|.$$

This completes the proof. ∎

THEOREM 4.3: *Let G = HN and let B be a p-block of G with a non-cyclic defect group. Then the number of B-weights is the number of irreducible Brauer characters of B.*

Proof: If $B = B_0$, then Theorem 4.3 follows by Lemmas 4.1, 4.2 and (4.1). If D is cyclic or SD_{16}, then Theorem 4.3 follows by [12, Theorem 7.11] and [21].

If $p = 3$ and $B = B_1$, then $D(B) = 3^2$, $N(D)/C(D)D \simeq 4.2^2 = SD_{16}$ has exactly 7 irreducible characters, so that $\mathcal{W}(B) = 7$. ∎

5. Radical chains of HN

Let G = HN, $C \in \mathcal{R}(G)$ and $N(C) = N_G(C)$. In this section we do some cancellations in the alternating sum of the refined Dade's conjecture. We first list some radical p-chains $C(i)$ and their normalizers for certain integers i, then reduce the proof of the conjecture to the subfamily $\mathcal{R}^0(G)$ of $\mathcal{R}(G)$, where $\mathcal{R}^0(G)$ is the union of G-orbits of all $C(i)$. The subgroups of the p-chains in Tables 3–5 are given either by Proposition 4.1 or by its proof. Moreover, if $E = \mathrm{Aut}(G)$, then $N_E(C(i)) = N_G(C(i)).2$ for each $C(i)$.

LEMMA 5.1: *Let $\mathcal{R}^0(G)$ be the G-invariant subfamily of $\mathcal{R}(G)$ such that*

$$\mathcal{R}^0(G)/G = \begin{cases} \{C(i) : 1 \le i \le 6\} & \text{with } C(i) \text{ given in Table 3 if } p = 5, \\ \{C(i) : 1 \le i \le 12\} & \text{with } C(i) \text{ given in Table 4 if } p = 3, \\ \{C(i) : 1 \le i \le 32\} & \text{with } C(i) \text{ given in Table 5 if } p = 2. \end{cases}$$

Then

(5.1)
$$\sum_{C\in\mathcal{R}(G)/G} (-1)^{|C|}\mathrm{k}(N(C), B_0, d, u, [r]) = \sum_{C\in\mathcal{R}^0(G)/G} (-1)^{|C|}\mathrm{k}(N(C), B_0, d, u, [r])$$

for all integers $d, u, r \ge 0$.

C		$N(C)$
$C(1)$	1	HN
$C(2)$	$1 < 5$	$(D_{10} \times U_3(5)).2$
$C(3)$	$1 < 5 < 5 \times 5_+^{1+2}$	$(D_{10} \times 5_+^{1+2}{:}\, 8).2$
$C(4)$	$1 < 5^2.5_+^{1+2}$	$5^2.5_+^{1+2} : 4A_5$
$C(5)$	$1 < 5_+^{1+4} < Sy_5$	$Sy_5.(2 \times 4)$
$C(6)$	$1 < 5_+^{1+4}$	$5_+^{1+4} : 2_-^{1+4}.5.4$

Table 3. Some radical 5-chains of HN

Proof: Case (1). Suppose p is odd and C' is a radical chain such that

$$C' : 1 < P_1' < \cdots < P_m'. \tag{5.2}$$

Let $C \in \mathcal{R}(G)$ be given by (2.1) with $P_1 \in \Phi(G,p)$. If $p = 5$, then $C' : 1 < 5^2.5_+^{1+2} < Sy_5$ and $g(C') : 1 < Sy_5$ have the same normalizers $N(C') = N(g(C')) = Sy_5.(2 \times 4)$ and $N_E(C') = N_E(g(C')) = N(C').2$, so that for any $B \in \mathrm{Blk}(G)$ and integers $d, u, r \ge 0$,

$$\mathrm{k}(N(C'), B, d, u, [r]) = \mathrm{k}(N(g(C')), B, d, u, [r]) \tag{5.3}$$

and we may suppose $C \neq_{\mathrm{HN}} C'$ and $g(C')$. The remaining chains are given by Table 3.

Suppose $p = 3$. Let $R \in \{3^4.3, Sy_3\} \subsetneq \Phi(3^4{:}\, 2(A_4 \times A_4).4, 3)$, and let $\sigma(R) : 1 < 3^4 < R$ with $Q = 3^4$, so that $\sigma(R)' : 1 < R$. Then $\sigma(R)$ and $\sigma(R)'$ are radical chains of HN satisfying the conditions of Lemma 2.2. Thus there is a bijection g from $\mathcal{R}^-(\sigma(R), 3^4)$ onto $\mathcal{R}^0(\sigma(R), 3^4)$ such that $N(C') = N(g(C'))$ and $N_E(C') = N_E(g(C')) = N(C').2$ for any $C' \in \mathcal{R}^-(\sigma(R), 3^4)$, so that (5.3) holds and we may suppose

$$C \notin \bigcup_{R \in \{3^4.3, Sy_3\}} (\mathcal{R}^-(\sigma(R), 3^4) \cup \mathcal{R}^0(\sigma(R), 3^4)).$$

Thus $P_1 \notin \{3^4.3, Sy_3\}$, and if $P_1 = 3^4$, then $C =_G C(2)$. We may suppose

$$P_1 \in \{3, 3^2, 3_+^{1+4}\} \subseteq \Phi(\mathrm{HN}, 3).$$

Let $C' : 1 < 3 < 3 \times S'$ and $g(C') : 1 < 3 < 3^4 < 3 \times S'$. As shown in the proof (2) of Proposition 4.1, $N(C') = N(g(C')) = (3 \times S').2^2$ and $N_E(C') = N_E(g(C')) = N(C').2$, so that (5.3) holds and we may suppose $C \neq_G C'$ and $g(C')$. Thus if $P_1 = 3$, then $C \in_{\mathrm{HN}} \{C(i) : 5 \leq i \leq 10\}$. If $P_1 = 3_+^{1+4}$, then $C \in_{\mathrm{HN}} \{C(3), C(4)\}$, and if $P_1 = 3^2$, then $C \in_{\mathrm{HN}} \{C(11), C(12)\}$.

C		$N(C)$
$C(1)$	1	HN
$C(2)$	$1 < 3^4$	$3^4\colon 2(A_4 \times A_4).4$
$C(3)$	$1 < 3_+^{1+4} < Sy_3$	$Sy_3.(2 \times 4)$
$C(4)$	$1 < 3_+^{1+4}$	$3_+^{1+4} : 4A_5$
$C(5)$	$1 < 3 < 3^2$	$3.(S_3 \times S_6)$
$C(6)$	$1 < 3$	$(3 \times A_9)\colon 2$
$C(7)$	$1 < 3 < 3^3$	$3^3\colon 2S_4$
$C(8)$	$1 < 3 < 3^2 < 3^4$	$3^4.(2 \times D_8)$
$C(9)$	$1 < 3 < 3^4$	$3^4.(2 \times S_4)$
$C(10)$	$1 < 3 < 3^3 < 3^3.3$	$3^3.3.2^2$
$C(11)$	$1 < 3^2 < 3^4$	$3^4.4^2.2^2$
$C(12)$	$1 < 3^2$	$(3^2 : 4 \times A_6).2^2$

Table 4. Some radical 3-chains of HN

Case (2). Suppose $p = 2$. We first consider the radical subgroups of G contained in M_2. Let $R \in \Phi(M_2, 2) \backslash \{2_+^{1+8}\}$ and let $\sigma(R) : 1 < Q = 2_+^{1+8} < R$, so that $\sigma(R)' : 1 < R$. A similar proof to that in the case $p = 3$ shows that we may suppose

$$C \notin \bigcup_{R \in \Phi(M_2, 2) \backslash \{2_+^{1+8}\}} (\mathcal{R}^-(\sigma(R), 2_+^{1+8}) \cup \mathcal{R}^0(\sigma(R), 2_+^{1+8})),$$

C		$N(C)$
$C(1)$	1	HN
$C(2)$	$1 < 2_+^{1+8}$	$2_+^{1+8}.(A_5 \times A_5){:}\ 2$
$C(3)$	$1 < 2^6 < 2^6.2^4$	$2^6.2^4.A_5$
$C(4)$	$1 < 2^6$	$2^6.U_4(2)$
$C(5)$	$1 < 2^6 < 2^2.2^3.2^6$	$2^2.2^3.2^6.3^2.2$
$C(6)$	$1 < 2^6 < 2^6.2^4 < 2^2.2^3.2^6.2$	$2^2.2^3.2^6.2.3$
$C(7)$	$1 < 2^3.2^2.2^6 < 2_+^{1+8}.2^4$	$2_+^{1+8}.2^4.(3 \times S_3)$
$C(8)$	$1 < 2^3.2^2.2^6$	$2^3.2^2.2^6.(L_3(2) \times 3)$
$C(9)$	$1 < 2 < 2^6$	$2^6.S_6$
$C(10)$	$1 < 2 < 2^6 < 2^6.2^3$	$2^6.2^3.S_3$
$C(11)$	$1 < 2 < 2^2.2^6$	$2^2.2^6.S_5$
$C(12)$	$1 < 2 < 2^2.2^6 < 2^2.2^4.2^3$	$2^2.2^4.2^3.S_3$
$C(13)$	$1 < 2 < 2^2.2^6 < 2^2.2^4.2^3 < 2^2.2^4.2^3.2$	$2^2.2^4.2^3.2$
$C(14)$	$1 < 2 < 2^4.2^4 < 2^2.2^3.2^5$	$2^2.2^3.2^5.S_3$
$C(15)$	$1 < 2 < 2^4.2^4$	$2^4.2^4.L_3(2)$
$C(16)$	$1 < 2$	$2.HS.2$
$C(17)$	$1 < 2 < 2^2$	$(2^2 \times A_8).2$
$C(18)$	$1 < 2 < 2^2 < 2^5$	$2^5 : L_3(2)$
$C(19)$	$1 < 2 < 2^2 < 2^5 < 2^4.2^3$	$2^4.2^3.S_3$
$C(20)$	$1 < 2 < 2^2 < 2^5 < 2^4.2^3 < 2^5.D_8$	$2^5.D_8$
$C(21)$	$1 < 2 < 2^2 < 2^5 < 2^3.2^4$	$2^3.2^4.S_3$
$C(22)$	$1 < 2 < 2^2 < 2^6$	$2^6.(S_3 \times S_3).2$
$C(23)$	$1 < 2 < 2^2 < 2^6 < 2^2.2^3.2^3.2$	$2^2.2^3.2^3.2$
$C(24)$	$1 < 2^2$	$(A_4 \times A_8).2$
$C(25)$	$1 < 2^2 < 2^5$	$A_4 \times 2^3 : L_3(2)$
$C(26)$	$1 < 2^2 < 2^5 < 2^3.2^4$	$2^3.2^4.(3 \times S_3)$
$C(27)$	$1 < 2^2 < 2^6$	$2^6.(3 \times S_3 \times S_3).2$
$C(28)$	$1 < 2^2 < 2^6 < 2^4.2^3$	$2^4.2^3.3.S_3$
$C(29)$	$1 < 2^2 < 2^6 < 2^4.2^3 < 2^4.2^3.2$	$2^4.2^3.2.3$
$C(30)$	$1 < 2^2 < 2^3.2^4 < 2^2.2^3.2^3$	$2^2.2^3.2^3.S_3$
$C(31)$	$1 < 2^2 < 2^3.2^4$	$2^3.2^4.(S_3 \times S_3)$
$C(32)$	$1 < 2^2 < 2^3.2^4 < 2^3.2^2.2^3$	$2^3.2^2.2^3.S_3$

Table 5. Some radical 2-chains of HN

so $P_1 \notin_G \{2_+^{1+8}.2, 2_+^{1+8}.2^2, 2_+^{1+8}.2^4, Sy_2\}$, and if $P_1 = 2_+^{1+8}$, then $C =_G C(2)$

Thus

$$P_1 \in_G \{2, 2^2, D_8, Q_8, SD_{16}, 2^6, 2^3.2^2.2^6, 2^2.2.2^2.2^4.2^4\}.$$

Case (2.1). Let $\sigma : 1 < Q = 2^3.2^2.2^6 < 2^2.2.2^2.2^4.2^4$, so that $\sigma' : 1 < 2^2.2.2^2.2^4.2^4$. A similar proof shows that we may suppose

$$C \notin (\mathcal{R}^-(\sigma, 2^3.2^2.2^6) \cup \mathcal{R}^0(\sigma, 2^3.2^2.2^6)).$$

Let $C' : 1 < 2^3.2^2.2^6 < Sy_2$ and $g(C') : 1 < 2^3.2^2.2^6 < 2_+^{1+8}.2^4 < Sy_2$. Then $N(C') = N(g(C'))$, $N_E(C') = N_E(g(C'))$ and we may delete C' and $g(C')$. Thus may suppose $P_1 \neq_G 2^2.2.2^2.2^4.2^4$ and, if $P_1 = 2^3.2^2.2^6$, then $C =_G C(7)$ or $C(8)$.

Case (2.2). If $S'' \in \Phi(M_3, 2)$, then by the proof (3.3) of Proposition 4.1, we may suppose $S'' \in \Phi(N_{M_3}(2^2.2^3.2^6), 2)$, and moreover, $N_{M_3}(S'') = N_{N_{M_3}(2^2.2^3.2^6)}(S'')$ and $N_{M_3.2}(S'') = N_{N_{M_3.2}(2^2.2^3.2^6)}(S'')$, where $M_3.2 = N_E(M_3)$. Let $\sigma : 1 < 2^6 < Q = 2^2.2^3.2^6 < S''$, so that $\sigma' : 1 < 2^6 < S''$. Then σ and σ' satisfy the conditions of Lemma 2.2. A similar proof to Case (1) shows that we may suppose

$$C \notin (\mathcal{R}^-(\sigma, 2^2.2^3.2^6) \cup \mathcal{R}^0(\sigma, 2^2.2^3.2^6)).$$

It follows that if $P_1 = 2^6$, then we may assume that

$$C \in_G \{C(3), C(4), C(5), C(6)\}.$$

Case (2.3). Let $R \in \{D_8, Q_8, SD_{16}\} \subseteq \Phi(M_1, 2) \cap \Phi(G, 2)$ and let $\sigma(R) : 1 < Q = 2 < R$. A similar proof shows that we may suppose

$$C \notin (\mathcal{R}^-(\sigma(R), 2) \cup \mathcal{R}^0(\sigma(R), 2)),$$

so $P_1 \neq_G R$, and if $P_1 = 2$, then $P_2 \neq_G R$. In particular, we may suppose

$$P_2 \in_G \{2^2, 2^6, 2^2.2^6, 2^4.2^4, 2^2.2^4.2^3, 2^2.2^3.2^5, 2^3.2^3.2^4, S'\} \subseteq \Phi(M_1, 2).$$

Let $P \in \{2^6, 2^2.2^6, 2^4.2^4\} \subseteq \Phi(M_1, 2)$. Then $N_{M_1}(P)$ is given in the proof (3.1) of Proposition 4.1 and $N_E(M_1) = M_1.2$. We may take

$$\Phi(N_{M_1}(P), 2) = \begin{cases} \{2^6, 2^6.2^3, 2^2.2^4.2^3, 2^2.2^4.2^3.2\} & \text{if } P = 2^6, \\ \{2^2.2^6, 2^2.2^4.2^3, 2^2.2^3.2^5, S'\} & \text{if } P = 2^2.2^6, \\ \{2^4.2^4, 2^3.2^3.2^4, 2^2.2^3.2^5, S'\} & \text{if } P = 2^4.2^4, \end{cases}$$

and $N_{M_1}(W) = N_{N_{M_1}(P)}(W)$, $N_{M_1.2}(W) = N_{N_{M_1.2}(P)}(W)$ for each $W \in \Phi(N_{M_1}(P), 2)$, except when $W = 2^6.2^3$ or $2^2.2^4.2^3.2$. In the former case $N_{N_{M_1}(P)}(W) = 2^6.2^3.S_3$ and in the latter case $N_{N_{M_1}(P)}(W) = 2^2.2^4.2^3.2$.

Let $R \in \Phi(N_{M_1}(Q), 2)$ such that $R = 2^2.2^4.2^3$ when $Q = 2^6$, $R = 2^2.2^3.2^5$ or S' when $Q = 2^2.2^6$, and $R = 2^3.2^3.2^4$ when $Q = 2^4.2^4$. In addition, let $\sigma(R) : 1 < 2 < Q < R$, so that $\sigma'(R) : 1 < 2 < R$. A similar proof shows that we may suppose

$$C \notin (\mathcal{R}^-(\sigma(R), Q) \cup \mathcal{R}^0(\sigma(R), Q)). \tag{5.4}$$

Thus we may suppose $P_2 \neq_G R$ and, if $P_2 = Q$, then $P_3 \neq_G R$.

Let $C' : 1 < 2 < 2^6 < 2^2.2^4.2^3.2$ and $g(C') : 1 < 2 < 2^6 < 2^6.2^3 < 2^2.2^4.2^3.2$. Then $N(C') = N(g(C'))$, $N_E(C') = N_E(g(C'))$ and we may delete C' and $g(C')$. Similarly, we may delete $C' : 1 < 2 < 2^4.2^4 < S'$ and $g(C') : 1 < 2 < 2^4.2^4 < 2^2.2^3.2^5 < S'$.

Thus if $P_1 = 2$ and $P_2 \neq_G 2^2$, then $C \in_G \{C(j) : 9 \leq j \leq 16\}$.

Case (2.4). Let $P_1 = 2$ and $P_2 = 2^2$. Then $N_{M_1}(2^2) = (2^2 \times A_8).2$ and $N_{M_1.2}(2^2) = N_{M_1}(2^2).2$. We may take

$$\Phi((2^2 \times A_8).2, 2) = \{2^2, D_8, 2^5, 2^6, 2^4.2^3, (2^4.2^3)^*, 2^2.2^3.2^3, 2^2.2^3.2^3.2\} \subseteq \Phi(M_5, 2),$$

and moreover, for $R \in \Phi((2^2 \times A_8).2, 2)$,

$$N_{M_1}(R) = \begin{cases} D_8 \times S_6 & \text{if } R = D_8, \\ 2^5 : L_3(2) & \text{if } R = 2^5, \\ 2^6.(S_3 \times S_3).2 & \text{if } R = 2^6, \\ 2^4.2^3.S_3 & \text{if } R = 2^4.2^3, \\ (2^4.2^3)^*.S_3 & \text{if } R = (2^4.2^3)^*, \\ 2^2.2^3.2^3.S_3 & \text{if } R = 2^2.2^3.2^3, \\ 2^2.2^3.2^3.2 & \text{if } R = 2^2.2^3.2^3.2, \end{cases}$$

and $N_{M_1.2}(R) = N_{M_1}(R).2$. In particular, if

$$W \in \{D_8, (2^4.2^3)^*, 2^2.2^3.2^3, 2^2.2^3.2^3.2\},$$

then $N_{M_5}(W) = N_{(2^2 \times A_8).2}(W)$ and $N_{M_5.2}(W) = N_{M_5}(W).2$. Let $\sigma(W) : 1 < Q = 2 < 2^2 < W$, so that $\sigma' : 1 < 2^2 < W$. A similar proof to Case (1) shows that we may suppose

$$C \notin (\mathcal{R}^-(\sigma(W), 2) \cup \mathcal{R}^0(\sigma(W), 2)).$$

It follows that if $P_1 = 2$ and $P_2 = 2^2$, then we may assume $P_3 \neq_G W$, and if $P_1 = 2^2$, then we may suppose $P_2 \neq_G W$.

Let $P \in \{2^5, 2^6\} \subseteq \Phi((2^2 \times A_8).2, 2)$. We may take

$$\Phi(N_{(2^2 \times A_8).2}(P), 2) = \begin{cases} \{2^5, 2^4.2^3, 2^3.2^4, 2^5.D_8\} & \text{if } P = 2^5, \\ \{2^6, 2^4.2^3, (2^4.2^3)^*, 2^2.2^3.2^3.2\} & \text{if } P = 2^6, \end{cases}$$

and $N_{(2^2\times A_8).2}(W) = N_{N_{(2^2\times A_8).2}(P)}(W)$ for each

$$W \in \{2^4.2^3, (2^4.2^3)^*, 2^2.2^3.2^3.2\};$$

in addition, $N_{N_{(2^2\times A_8).2}(2^5)}(2^3.2^4) = 2^3.3^4.S_3$.

Let $C' : 1 < 2 < 2^2 < 2^5 < 2^5.D_8$ and $g(C') : 1 < 2 < 2^2 < 2^5 < 2^3.2^4 < 2^5.D_8$. Then $N(C') = N(g(C')) = 2^5.D_8$, $N_E(C') = N_E(g(C')) = 2^5.D_8.2$ and we may delete C' and $g(C')$.

Let $\sigma : 1 < 2 < 2^2 < Q = 2^6 < 2^4.2^3$, so that $\sigma' : 1 < 2 < 2^2 < 2^4.2^3$ and σ, σ' satisfy the conditions of Lemma 2.2. A similar proof to Case (1) shows that we may suppose

$$C \notin (\mathcal{R}^-(\sigma, 2^6) \cup \mathcal{R}^0(\sigma, 2^6)).$$

Thus if $P_1 = 2$ and $P_2 = 2^2$, then $C \in \{C(j) : 17 \leq j \leq 23\}$.

Case (2.5). Let $P_1 = 2^2$ and $P \in \{2^5, 2^6, 2^3.2^4\} \subseteq \Phi(M_5, 2)$. Then $N_{M_5}(P)$ is given in the proof (3.5) of Proposition 4.1 and $N_E(M_5) = M_5.2$. We may take

$$\Phi(N_{M_5}(P), 2) = \begin{cases} \{2^5, 2^3.2^4, 2^4.2^3, 2^5.D_8\} & \text{if } P = 2^5, \\ \{2^6, 2^4.2^3, (2^4.2^3)^*, 2^3.2^2.2^3, 2^2.2^3.2^3.2\} & \text{if } P = 2^6, \\ \{2^3.2^4, 2^2.2^3.2^3, 2^3.2^2.2^3, 2^2.2^3.2^3.2\} & \text{if } P = 2^3.2^4, \end{cases}$$

and $N_{M_5}(W) = N_{N_{M_5}(P)}(W)$, $N_{M_5.2}(W) = N_{N_{M_5.2}(P)}(W)$ for each $W \in \Phi(N_{M_5}(P), 2)$, except when $W = 2^3.2^4$ or $2^5.D_8$. In the former case $N_{N_{M_5}(P)}(W) = 2^3.2^4.(3 \times S_3)$ and in the latter case $N_{N_{M_5}(P)}(W) = 2^5.D_8$.

Let $R \in \Phi(N_{M_5}(Q), 2)$ such that $R = 2^4.2^3$ when $Q = 2^5$, and $R = 2^3.2^2.2^3$ when $Q = 2^6$. In addition, let $\sigma(R) : 1 < 2^2 < Q < R$, so that $\sigma' : 1 < 2^2 < R$. A similar proof shows that we may suppose (5.4) holds. Thus we may suppose $P_2 \neq_G Q$ and, if $P_2 = P$, then $P_3 \neq_G Q$. We may suppose

$$P_2 \in_G \{2^5, 2^6, 2^3.2^4\} \subseteq \Phi(M_5, 2).$$

Case (2.6). Let $\sigma : 1 < Q = 2 < 2^2 < 2^6 < (2^4.2^3)^*$, so that $\sigma' : 1 < 2^2 < 2^6 < (2^4.2^3)^*$. A similar proof to Case (1) shows that we may suppose

$$C \notin (\mathcal{R}^-(\sigma, 2) \cup \mathcal{R}^0(\sigma, 2)).$$

Let $C' : 1 < 2^2 < 2^5 < 2^5.D_8$ and $g(C') : 1 < 2^2 < 2^5 < 2^3.2^4 < 2^5.D_8$. Then $N(C') = N(g(C')) = 2^5.D_8$, $N_E(C') = N_E(g(C')) = 2^5.D_8.2$ and we may delete C' and $g(C')$. Similarly, we may delete $C' : 1 < 2^2 < 2^6 < 2^2.2^3.2^3.2$, $g(C') : 1 < 2^2 < 2^3.2^4 < 2^2.2^3.2^3 < 2^2.2^3.2^4.2$, and $C' : 1 < 2^2 < 2^3.2^4 < 2^2.2^3.2^3.2$, $g(C') : 1 < 2^2 < 2^3.2^4 < 2^3.2^2.2^3 < 2^2.2^3.2^4.2$. Thus if $P_1 = 2^2$, then $C \in \{C(i) : 24 \leq i \leq 32\}$. ∎

6. The proof of Uno's version of Dade's conjecture

Let $L = N(C)$ or $N_E(C)$ be the normalizer of a radical p-chain. If L is a maximal subgroup of HN or $E = \mathrm{Aut}(\mathrm{HN})$, then the character table of L can be found in the library of character tables distributed with GAP [14] except when $L = 4.HS.2$. We calculate the degrees of the irreducible characters of $4.HS.2$ as follows: first for each maximal 2-local subgroup M of $N_E(C(16))$, determine the fusions of conjugacy classes of M in $N_E(M) = M.2$; then get the fusions of conjugacy classes of $N_{\mathrm{HN}}(C(16)) = 2.HS.2$ in $4.HS.2$; and finally obtain the fusions of irreducible characters of $2.HS.2$ using Brauer's permutation lemma (cf. [18, Lemma 3.2.19]). The character table of $2.HS.2$ is in the library of character tables distributed with GAP.

The tables listing degrees of irreducible characters referenced in the proof of Theorem 6.1 are available electronically [7].

THEOREM 6.1: *Let B be a p-block of $G = \mathrm{HN}$ with a positive defect. Then B satisfies Uno's refinement of the invariant conjecture of Dade.*

Proof: By Proposition 2.1 and [21], we may suppose $D(B)$ is non-cyclic and $D(B) \not\simeq SD_{16}$, so that $B = B_0$ is principal, except when $p = 3$, in which case $B = B_0$ or B_1. We set $\mathrm{k}(\ell, d, u, r) = \mathrm{k}(N(C(\ell)), B_0, d, u, [r])$ and $\mathrm{k}(\ell, d, u) = \mathrm{k}(N(C(\ell)), B_0, d, u)$ for integers ℓ, d, u and $1 \le r \le (p-1)/2$.

Case (1). Suppose $p = 5$. The values $\mathrm{k}(i, d, u, r)$ are given in Table 6.

Defect d	6	6	5	5	5	5	4	4	4	4	3	otherwise
Value u	2	1	2	2	1	1	2	2	1	1	2	otherwise
Value r	2	1	2	1	2	1	2	1	2	1	2	otherwise
k(1, d, u, r)	10	10	10	4	4	2	0	0	0	2	3	0
k(2,d,u,r)=k(3,d,u,r)	0	0	0	0	0	0	22	9	4	2	10	0
k(4, d, u, r)	10	10	4	0	16	18	0	0	0	2	0	0
k(5, d, u, r)	10	10	4	0	16	18	2	0	8	0	0	0
k(6, d, u, r)	10	10	10	4	4	2	2	0	8	0	3	0

Table 6. Values of $\mathrm{k}(i, d, u, r)$ when $p = 5$

It follows that

$$\sum_{i=1}^{6} (-1)^{|C(i)|} \mathrm{k}(N(C(i)), B_0, d, u, [r]) = 0$$

and Theorem 6.1 follows. Since $\mathrm{k}(1, 6, u, r) = \mathrm{k}(5, 6, u, r) = 10$ and since $N_G(C(5)) = N_G(Sy_5)$, it follows that the Isaacs–Navarro conjecture holds for B, which already follows by [17].

Case (2). Suppose $p = 3$. Let $B = B_1$ and

$$\mathrm{Blk}(N(C(i))|B) = \{b \in \mathrm{Blk}(N(C(i))) : b^G = B\}.$$

Then $\mathrm{Blk}(N(C(i))|B) = \emptyset$ except when $i \in \{1, 5, 6, 12\}$; in these cases

$$\mathrm{k}(N(C(i)), B_1, d, u) = \begin{cases} 9 & \text{if } d = 2 \text{ and } u = 2, \\ 0 & \text{otherwise.} \end{cases}$$

This proves the theorem when $B = B_1$.

Suppose $B = B_0$. The values $\mathrm{k}(i, d, u)$ are given in Table 7.

Defect d	6	6	5	5	4	4	3	3	otherwise
Value u	2	1	2	1	2	1	2	1	otherwise
k(1, d, u) = k(4, d, u)	12	6	0	6	4	2	1	2	0
k(2, d, u) = k(3, d, u)	12	6	6	0	4	2	0	0	0
k(5, d, u) = k(8, d, u)	0	0	0	0	40	2	0	0	0
k(6, d, u) = k(9, d, u)	0	0	13	2	19	2	0	0	0
k(7, d, u) = k(10, d, u)	0	0	0	0	12	6	3	0	0
k(11, d, u) = k(12, d, u)	0	0	0	0	25	8	0	0	0

Table 7. Values of $\mathrm{k}(i, d, u)$ when $p = 3$

It follows that

$$\sum_{i=1}^{12} (-1)^{|C(i)|} \mathrm{k}(N(C(i)), B_0, d, u) = 0$$

and Theorem 6.1 follows.

Case (3). Suppose $p = 2$. We first consider the chains C with $\mathrm{d}(N(C)) = 8$, so that $C \in \{C(18), C(19), C(20), C(21), C(25), C(26), C(28), C(29)\}$. Then $N(C)$ has only the principal block and the values $\mathrm{k}(i, d, u)$ are given in Table 8.

Defect d	8	8	7	7	6	6	5	5	otherwise
Value u	2	1	2	1	2	1	2	1	otherwise
k(18, d, u) = k(25, d, u)	16	16	4	4	0	0	2	2	0
k(19, d, u) = k(28, d, u)	16	16	12	12	0	0	0	0	0
k(20, d, u) = k(29, d, u)	16	16	12	12	4	4	0	0	0
k(21, d, u) = k(26, d, u)	16	16	4	4	4	4	2	2	0

Table 8. Values of $\mathrm{k}(i, d, u)$ when $p = 2$ and $\mathrm{d}(N(C(i))) = 8$

It follows that

$$\sum_{\mathrm{d}(N(C))=8} (-1)^{|C|}\mathrm{k}(N(C), B_0, d, u) = 0.$$

Next we consider the chains C such that $\mathrm{d}(N(C)) = 9$, so that

$$C \in \{C(17), C(22), C(23), C(24), C(27), C(30), C(31), C(32)\}.$$

Then $N(C)$ has only the principal block except when $C = C(17)$ or $C(24)$, in which cases $N(C)$ contains exactly two blocks b_0 and b_1 such that each $b_j^G = B_j$, and

$$\mathrm{k}(N(C(17)), B_1, d, u) = \mathrm{k}(N(C(24)), B_1, d, u) = \begin{cases} 4 & \text{if } d = 3 \text{ and } u = 2, \\ 1 & \text{if } d = 2 \text{ and } u = 2, \\ 0 & \text{otherwise.} \end{cases}$$

The values $\mathrm{k}(i, d, u)$ are given in Table 9 with $l \in \{17, 24, 30, 31\}$ and $j \in \{22, 23, 27, 32\}$.

Defect d	9	8	8	7	7	6	5	otherwise
Value u	2	2	1	2	1	2	2	otherwise
k(l, d, u)	16	16	4	10	0	6	1	0
k(j, d, u)	16	16	4	14	4	2	0	0

Table 9. Values of $\mathrm{k}(i, d, u)$ when $p = 2$ and $\mathrm{d}(N(C(i))) = 9$

It follows that

$$\sum_{\mathrm{d}(N(C))=9} (-1)^{|C|}\mathrm{k}(N(C), B_0, d, u) = 0.$$

Suppose $C = C(i)$ is a chain with $\mathrm{d}(N(C)) = 10$. Then

$$C \in \{C(9), C(10), C(12), C(13)\}$$

and $N(C)$ has only the principal block. The values $\mathrm{k}(i, d, u)$ are given in Table 10.

Defect d	10	9	8	8	7	7	6	otherwise
Value u	2	2	2	1	2	1	2	otherwise
k(9, d, u)	32	8	4	4	0	4	4	0
k(10, d, u)	32	8	12	12	0	4	0	0
k(12, d, u)	32	24	4	4	8	0	4	0
k(13, d, u)	32	24	12	12	8	0	0	0

Table 10. Values of $\mathrm{k}(i, d, u)$ when $p = 2$ and $\mathrm{d}(N(C(i))) = 10$

It follows that

$$\sum_{d(N(C))=10} (-1)^{|C|} k(N(C), B_0, d, u) = 0.$$

Suppose $C = C(i)$ is a chain with $d(N(C)) = 11$. Then

$$C \in \{C(11), C(14), C(15), C(16)\}$$

and $N(C)$ has only the principal block except when $C = C(16)$, in which case $N(C(16))$ contains exactly two blocks b_0 and b_1 such that each $b_j^G = B_j$, and

$$k(N(C(16)), B_1, d, u) = \begin{cases} 4 & \text{if } d = 4 \text{ and } u = 2, \\ 1 & \text{if } d = 3 \text{ and } u = 2, \\ 2 & \text{if } d = 3 \text{ and } u = 1, \\ 1 & \text{if } d = 2 \text{ and } u = 2, \\ 0 & \text{otherwise.} \end{cases} \tag{6.1}$$

The values $k(i, d, u)$ are given in Table 11.

Defect d	11	10	9	8	8	7	6	otherwise
Value u	2	2	2	2	1	2	2	otherwise
k(11, d, u)	16	12	10	12	6	2	4	0
k(14, d, u)	16	12	10	12	6	2	0	0
k(15, d, u)	16	12	2	4	10	1	0	0
k(16, d, u)	16	12	2	4	10	1	4	0

Table 11. Values of $k(i, d, u)$ when $p = 2$ and $d(N(C(i))) = 11$

It follows that

$$\sum_{d(N(C))=11} (-1)^{|C|} k(N(C), B_0, d, u) = 0.$$

Now suppose $C = C(i)$ is a chain with $d(N(C)) = 12$. Then

$$C \in \{C(3), C(4), C(5), C(6)\}$$

and $N(C)$ has only the principal block. The values $k(i, d, u)$ are given in Table 12.

Defect d	12	12	11	11	10	10	9	9	8	8	6	otherwise
Value u	2	1	2	1	2	1	2	1	2	1	2	otherwise
k(3, d, u)	8	8	8	4	12	0	3	2	4	4	2	0
k(4, d, u)	8	8	4	4	12	0	3	2	0	0	2	0
k(5, d, u)	8	8	4	4	12	8	11	10	0	0	0	0
k(6, d, u)	8	8	8	4	12	8	11	10	4	4	0	0

Table 12. Values of $\mathrm{k}(i, d, u)$ when $p = 2$ and $\mathrm{d}(N(C(i))) = 12$

It follows that

$$\sum_{\mathrm{d}(N(C))=12} (-1)^{|C|}\mathrm{k}(N(C), B_0, d, u) = 0.$$

Finally, suppose $C = C(i)$ is a chain with $\mathrm{d}(N(C)) = 14$. Then

$$C \in \{C(1), C(2), C(7), C(8)\}$$

and $N(C)$ has only the principal block except when $C = C(1)$, in which case $N(C(1)) = G$ and $\mathrm{Irr}^+(G)$ consists of two blocks B_0 and B_1 given in Lemma 4.2 (c). In particular, $\mathrm{k}(N(C(1)), B_1, d, u)$ is the same as $\mathrm{k}(N(C(16)), B_1, d, u)$ given in (6.1). The non-zero values $\mathrm{k}(i, d, u)$ are given in Table 13.

Defect d	14	14	13	13	12	11	11	10	10	9	9	8	8	7	7	6	6
Value u	2	1	2	1	2	2	1	2	1	2	1	2	1	2	1	2	1
k(1,d,u)	8	8	4	4	0	7	4	4	0	1	0	0	0	0	0	3	2
k(2,d,u)	8	8	4	4	4	7	2	4	0	1	0	2	4	1	2	3	2
k(7,d,u)	8	8	4	4	4	7	2	4	8	1	2	2	4	1	2	0	0
k(8,d,u)	8	8	4	4	0	7	4	4	8	1	2	0	0	0	0	0	0

Table 13. Non-zero values of $\mathrm{k}(i, d, u)$ when $p = 2$ and $\mathrm{d}(N(C(i))) = 14$

It follows that

$$\sum_{\mathrm{d}(N(C))=14} (-1)^{|C|}\mathrm{k}(N(C), B_0, d, u) = 0$$

and Theorem 6.1 follows. ∎

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